

Ma 1113 Name: _____
 Midterm _____

Fall 2003
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① "Solve" the inequality $|5x+13| < 7$.

② Find the following limits:

③ $\lim_{x \rightarrow 0} \frac{\ln(1+ax)}{bx}$ ($a \neq 0, b \neq 0$)

④ $\lim_{x \rightarrow 17} \frac{x^2 - 5x - 204}{x - 17}$

⑤ $\lim_{x \rightarrow \pm\infty} \frac{39x^{17} + 3}{3x^{17} + 8x^4 + 1}$

⑥ $\lim_{x \rightarrow \pm\infty} \frac{x^7 + 7}{x^7} \frac{3x^3 + 2}{5x^3 + 49}$

⑦ $\lim_{x \rightarrow 0} \frac{\sin x}{7x^7 - x}$

⑧ $\lim_{x \rightarrow a^-} \frac{|x^2 - a^2|}{x - a}$

⑨ (a) Prove, from first principles,

that $f(x) := \sqrt[3]{x} := x^{1/3}, x \geq 0 \Rightarrow$

$f'(x) = \frac{1}{3x^{2/3}}, x > 0$ ($x^{2/3} := (x^2)^{1/3} = (x^{1/3})^2$).

③ ⑥ Find the equation of the tangent line to the curve $y = \sqrt[3]{x}$ at the point $(1, 1)$ on the curve.

④ Let $f(x) := \frac{1+x}{1-x}$.

a) What are the domain and range of f ?

b) What is f' and its domain and range? On what intervals is f increasing? Decreasing?

c) Find all asymptotes of f .
Graph f (roughly!).

d) What are $f \circ f$, $(f \circ f)'$ and f^{-1} , the inverse function of f .

(The composition of f and g is, $(f \circ g)(x) := f(g(x))$.)

⑤ Let $f(x) := \left(1 + \frac{1}{x}\right)^x$, $x \neq 0$. Show that: ① $f(x) \rightarrow e$, $x \rightarrow +\infty$
② $f(x) \rightarrow 1$, $x \rightarrow 0^+$

⑥ Evaluate

$$\lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right).$$

⑦ Prove, by induction, that

(a) $1 + z + z^2 + \dots + z^{n-1} = \frac{1-z^n}{1-z}, z \neq 1$
 $= n, z = 1$

- (b) $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$

(c) $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+\frac{1}{2})(n+1)}{3}$

(d) $1^3 + 2^3 + 3^3 + \dots + n^3 = (1+2+3+\dots+n)^2$

(e) $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

(In all of the above, $n=0, 1, 2, \dots !$)

Points 1. 10

2. 20

3. 20

4. 20

5. 20

6. 20

7. 20

130

Math 1113 Midterm Solutions

Fall 2003

Grassy

$$\textcircled{1} \quad \text{"Solve"} \quad |5x+13| < 7.$$

$$\text{II} \quad \text{Same as} \quad -7 < 5x+13 < 7$$

$$\text{“ “ } -20 < 5x < -6$$

$$\text{“ “ } \boxed{-4 < x < -\frac{6}{5}}$$

\textcircled{2} Find limits.

\textcircled{a} $a \neq 0, b \neq 0$

$$\text{Known} \quad \frac{\ln(1+x)}{x} \rightarrow 1, \quad x \rightarrow 0$$

(by L'Hospital, or a basic inequality).

So

$$\frac{\ln(1+ax)}{bx} = \frac{\ln(1+ax)}{ax} \cdot \frac{a}{b}$$

$$\rightarrow 1 \cdot \frac{a}{b} = \frac{a}{b} \quad \text{as } x \rightarrow 0$$

$$\Leftrightarrow ax \rightarrow 0$$

↑

since $a \neq 0$

\textcircled{b} $x \rightarrow 17$

$$\frac{x^2 - 5x - 204}{x - 17} = x + 12 \quad (x \neq 17)$$

$$x - 17 \frac{x + 12}{x^2 - 5x - 204} \rightarrow 17 + 12 = \boxed{29}, \quad x \rightarrow 17$$

$$\begin{array}{r} 17 \\ 12 \\ \hline 34 \\ 17 \\ \hline 204 \end{array}$$

\textcircled{c} $x \rightarrow \pm \infty$

$$\frac{39x^{17} + 3}{3x^{17} + 8x^4 + 1} = \frac{39 \left(1 + \frac{3}{39x^{17}}\right)}{3 \left(1 + \frac{8}{3x^{17}} + \frac{1}{3x^{17}}\right)} \rightarrow \boxed{13}$$

(2) d) $\lim_{x \rightarrow \pm\infty} f(x)$

$$\frac{x^7 + 7}{x^7} \cdot \frac{3x^3 + 2}{5x^3 + 49} = \left(1 + \frac{7}{x^7}\right) \frac{3x^3}{5x^3} \frac{\left(1 + \frac{2}{3x^3}\right)}{\left(1 + \frac{49}{5x^3}\right)}$$

$$\rightarrow 1 \quad \rightarrow 1$$

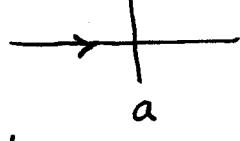
$$\rightarrow \boxed{\frac{3}{5}}$$

e) $x \rightarrow 0$

$$\frac{\sin x}{7x^2 - x} = \underbrace{\frac{\sin x}{x}}_{\rightarrow 1} \cdot \frac{1}{\frac{7x^2 - 1}{x} \rightarrow -1}$$

$$\rightarrow 1 \text{ by "fundamental trig inequality"}$$

$$\rightarrow \boxed{-1}$$

f) $x \rightarrow a^- = a - 0$ 
 means $x \rightarrow a$ from the Left!

$$\frac{|x^2 - a|}{x - a} = \underbrace{\frac{|x-a|}{x-a}}_{\rightarrow |2a|} \underbrace{|x+a|}_{\rightarrow |2a|} \rightarrow \boxed{-2|a|}$$

$$x < a \quad \Rightarrow -1 \text{ for } x < a$$

means $x - a < 0$

(3) a) $f(x) := \sqrt[3]{x}, x \geq 0 \Rightarrow f'(x) = \frac{1}{3x^{2/3}}, x > 0$
 from 1st princ.

$\square x > 0 \Rightarrow x+h > 0$ if h small enough.

$$\frac{1}{h} [f(x+h) - f(x)] = \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} \cdot \frac{1}{\sqrt[3]{x+h}}$$

(3)

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

from geometric sums!

Take $a = \sqrt[3]{x+h}$, $b = \sqrt[3]{x}$, so

$$\begin{aligned} \sqrt[3]{x+h} - \sqrt[3]{x} &= \frac{a^3 - b^3}{a^2 + ab + b^2} \\ &= \frac{x+h-x}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}} \end{aligned}$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h} \frac{1}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}} \\ &\rightarrow \frac{1}{3x^{2/3}}, \quad h \rightarrow 0 \\ &\approx \frac{1}{3}x^{-\frac{2}{3}} \end{aligned}$$

(b) Tangent line to curve $y = x^{1/3}$
at point $(1, 1)$ on curve.

Check that $(1, 1)$ is on curve. Klar!

(all this $y = f(x) = x^{1/3}$, so $f(1) = 1$.

$y' = f'(x) = \frac{1}{3}x^{-2/3}$ was rigorously done
in (a), and $f'(1) = \frac{1}{3}$. The linearization
of f about $(1, 1)$ is the line

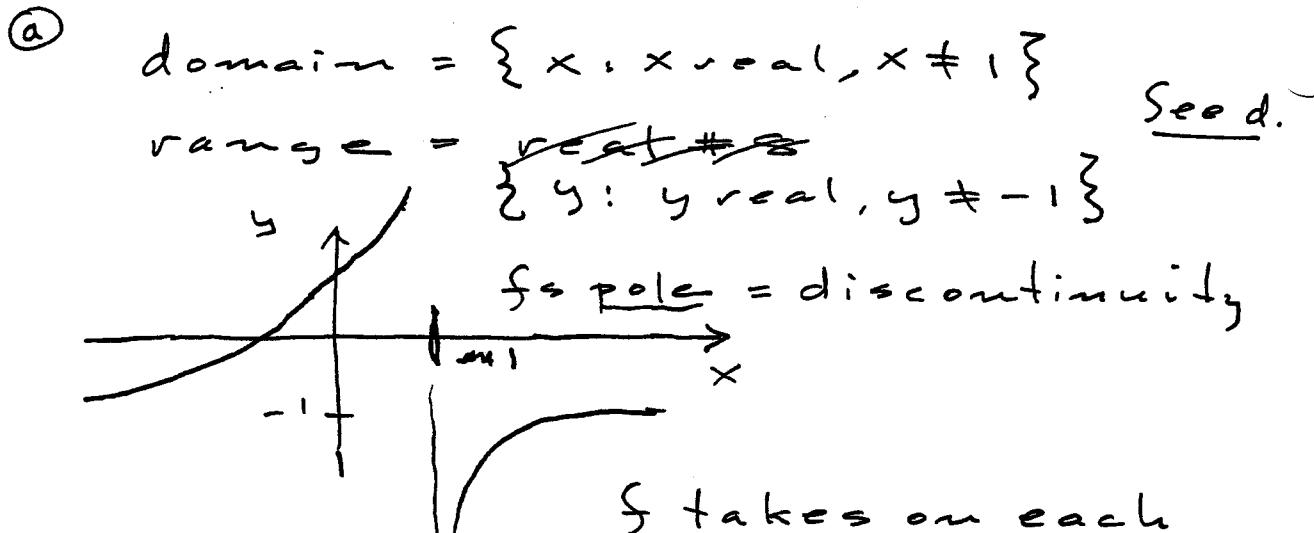
$$y = f(1) + f'(1)(x-1) = 1 + \frac{x-1}{3} = \frac{2}{3} + \frac{x}{3}$$

that's all there is to it.

$$y = \boxed{\frac{2+x}{3}}$$

(4)

$$\textcircled{4} \quad f(x) := \frac{1+x}{1-x}$$



f takes on each real #, exactly once, except -1.

$$\textcircled{b} \quad f'(x) = \frac{(1-x)(1-(1+x)(-1))}{(1-x)^2} = \frac{2}{(1-x)^2} > 0$$

$$f'(x) \rightarrow 0, x \rightarrow \pm \infty$$

$$\rightarrow +\infty, x \rightarrow +1$$

$\underbrace{x \neq +1}_{\text{domain}}$

range of $f = (0, +\infty)$, doubly open!

f strictly increases
everywhere on its domain,
but there's an infinite negative jump, from $+\infty$ to $-\infty$, as x passes thru +1, going from left to right!

(3)

c) 1 horizontal asymptote @ $y = -1$

1 vertical asymptote @ $x = +1$
See above graph.

$$\textcircled{d} \quad (f \circ f)(x) := f(f(x))$$

$$= \frac{1+f(x)}{1-f(x)} = \frac{1+\frac{1+x}{1-x}}{1-\frac{1+x}{1-x}}$$

The discontinuity

@ $x = 1$ got

removed by the
composition
process

$$\frac{\cancel{1-x+1+x}}{\cancel{1-x}-\cancel{x-x}} = \frac{2}{-2x}$$

$$= -\frac{1}{x}, \quad x \neq 0!$$

$$(f \circ f)'(x) = \frac{d}{dx} \left(-\frac{1}{x} \right) = \frac{1}{x^2}, \quad x \neq 0$$

$y = f(x) = \frac{1+x}{1-x}$ means that

$$y(1-x) = 1+x \quad (\text{when } x \neq 1)$$

that is

$$y - xy = 1 + x$$

that is

$$y - 1 = x(y + 1)$$

that is

$$x = \frac{y-1}{y+1} = f^{-1}(y), \quad y \neq -1$$

So

$$f^{-1}(x) = \frac{x-1}{x+1}, \quad x \neq -1$$

$$= -\frac{1}{f(x)} \quad (x \neq \pm 1).$$

(The inverse function is not the reciprocal!)

$$\textcircled{5} \quad f(x) = \left(1 + \frac{1}{x}\right)^x = e^{x \ln\left(1 + \frac{1}{x}\right)}, \quad x > 0$$

Always $x > 0$ in either limit,
and $\ln\left(1 + \frac{1}{x}\right)$ is defined there
since $x > 0 \Rightarrow 1 + \frac{1}{x} > 1$; in fact
 $x \ln\left(1 + \frac{1}{x}\right) > 0$ for $x > 0$.

\textcircled{a} As $x \rightarrow +\infty$, $\frac{1}{x} \rightarrow 0$, $1 + \frac{1}{x} \rightarrow 1$,
 $\ln\left(1 + \frac{1}{x}\right) \rightarrow 0$, $x \ln\left(1 + \frac{1}{x}\right)$ is " $\infty \cdot 0$ ",
an indeterminate form. Two ways
to rewrite this, as $\frac{0}{0}$ or $\frac{+\infty}{+\infty}$. Try
the first 1st

$$x \ln\left(1 + \frac{1}{x}\right) = \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \underset{x \rightarrow +\infty}{\sim} \frac{0}{0}$$

justifying L'H. Appl, L'H:

$$\frac{\frac{d}{dx} \ln\left(1 + \frac{1}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)} = \frac{\frac{1}{1 + \frac{1}{x}} \left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \frac{1}{1 + \frac{1}{x}}$$

$$\rightarrow 1, \quad x \rightarrow +\infty$$

Ans. $\left(1 + \frac{1}{x}\right)^x \rightarrow e^1 = e, \quad x \rightarrow +\infty$

(since e^y is continuous,
 $e^y \rightarrow e^1$ as $y \rightarrow 1$)

(5)

$$x \rightarrow 0^+ \Rightarrow f(x) \rightarrow ?$$

(7)

$$f(x) = e^{x \ln(1 + \frac{1}{x})}$$

$$x \ln(1 + \frac{1}{x}) \text{ is } 0 \cdot \ln(+\infty) = 0(+\infty)$$

indeterminate

rewrite as $\frac{0}{0}$ or $\frac{+\infty}{+\infty}$. "Try" the last first!

$$x \ln(1 + \frac{1}{x}) = \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} \text{ is } \frac{+\infty}{+\infty}$$

so can apply L'H: as before

$$\frac{\frac{d}{dx} \ln(1 + \frac{1}{x})}{\frac{d}{dx} \frac{1}{x}} = \frac{\frac{1}{1 + \frac{1}{x}}}{-\frac{1}{x^2}} = \frac{x}{1+x}$$

$$\rightarrow 0, x \rightarrow 0^+$$

so

$$x \ln(1 + \frac{1}{x}) \rightarrow 0, x \rightarrow 0^+$$

and

$$f(x) = e^{x \ln(1 + \frac{1}{x})} \rightarrow e^0 = 1,$$

$$x \rightarrow 0^+,$$

Since e^x is continuous in \mathbb{R} .

(9)

$$= \frac{\cancel{h^2}}{\cancel{h^2}} \frac{\frac{1}{2} - \frac{h}{3} + \dots}{1 - \frac{h}{2} + \dots}$$

$$\rightarrow \boxed{\frac{1}{2}}, h \rightarrow 0.$$

I did not write down the remainder terms because I did not (want to) know how many powers of h I would lose in the cancellation. But consider them there (hidden amongst the \dots s!). Taylor series are easier than many differentiations. "L'H" is a "tedious" process, as usually presented. Always try Taylor first, if you can. You might like it!

⑦ Induction problems.

① $P_n: \underbrace{1+z+z^2+\dots+z^{n-1}}_{\text{n terms}} = \frac{1-z^n}{1-z}, z \neq 1$

\square $= n$ if $z=1$

Done for all $n=0, 1, 2, \dots$, if $z=1$.

Drop $z=1$ from P_n .

(10)

□ P_0 is true $n=0 \Rightarrow 0$ terms

and $\sum n$ no term = 0 by natural convention, and $\frac{1-z^0}{1-z} = \frac{1-1}{1-z} = 0$ ($z \neq 1$). Assume P_n true, as stated (with $z \neq 1$ assumed).

Then

$$\begin{aligned} P_{n+1}: 1+z+\dots+z^n &= (1+z+\dots+z^{n-1}) + z^n \\ &= \frac{1-z^n}{1-z} + z^n = \frac{1-z^n}{1-z} + z^n \frac{1-z}{1-z} \\ &= \frac{\cancel{1-z^n} + \cancel{z^n} - z^{n+1}}{1-z} = \frac{1-z^{n+1}}{1-z}, \end{aligned}$$

is true. By induction, P_n is true for all $n=0, 1, 2, \dots$. ■

(b) $P_n: 1+2+3+\dots+n = \frac{n(n+1)}{2},$

$$n=0, 1, 2, \dots.$$

□ True for $n=0$: empty sum = $\frac{0 \cdot 1}{2} = 0$ ✓

Assume true for n , as stated. Then

$$\begin{aligned} P_{n+1}: 1+2+\dots+(n+1) &= (1+2+\dots+n)+(n+1) \\ &= \frac{n(n+1)}{2} + (n+1) = (n+1) \left(\frac{n}{2} + 1 \right) = \\ &= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}, \end{aligned}$$

as required; so P_{n+1} true $\Rightarrow P_n$ true for all $n=0, 1, 2, \dots$. ■

$$\textcircled{C} \quad P_m : 1^2 + 2^2 + \dots + n^2 = \frac{n(n+\frac{1}{2})(n+1)}{3},$$

□ True for $n=0$: empty sums!

Assume true for $n \geq 0$. Then

$$\begin{aligned} 1^2 + 2^2 + \dots + (n+1)^2 &= (1^2 + 2^2 + \dots + n^2) + (n+1)^2 = \\ &= \frac{n(n+\frac{1}{2})(n+1)}{3} + (n+1)^2 = \\ &= (n+1) \left(n+1 + \frac{n(n+\frac{1}{2})}{3} \right) \\ &= (n+1) \frac{3n+3+n^2+\frac{1}{2}n}{3} \\ &= (n+1) \frac{n^2+\frac{7}{2}n+3}{3} \\ &= (n+1) \frac{(n+\frac{3}{2})(n+2)}{3} \\ &= \frac{(n+1)((n+1)+\frac{1}{2})((n+1)+1)}{3}. \end{aligned}$$

$\hookrightarrow P_{n+1}$ true. By induction P_n true for $n=0, 1, 2, \dots$ ■

They're getting harder, so \textcircled{D} is a hint, and very helpful, for \textcircled{C} .

$$\textcircled{D} \quad 1^3 + 2^3 + \dots + n^3 = (1+2+\dots+n)^2.$$

□ P_0 is true, an "empty trivial assertion"! Assume P_n true, some

(12)

$n \geq 0$. Prove P_{n+1} true. Now

$$\begin{aligned} 1^3 + 2^3 + \cdots + (n+1)^3 &= \\ &= (1^3 + 2^3 + \cdots + n^3) + (n+1)^3 \\ &= (1+2+\cdots+n)^2 + (n+1)^3 \\ &= (1+2+\cdots+n)^2 + n(n+1)^2 + (n+1)^2 \end{aligned}$$

But by (b)

$$n(n+1) = 2(1+2+\cdots+n),$$

so

$$\begin{aligned} 1^3 + \cdots + (n+1)^3 &= \\ &= (1+2+\cdots+n)^2 + 2(1+2+\cdots+n)(n+1) + (n+1)^2 \\ &= (1+2+\cdots+n+(n+1))^2, \end{aligned}$$

since

$$a^2 + 2ab + b^2 = (a+b)^2.$$

So P_{n+1} is true. By induction P_n is true for all $n=0, 1, 2, \dots$. \blacksquare

(e) is now trivial, using (d)!

$$\begin{aligned} \square \quad 1^3 + 2^3 + \cdots + n^3 &= (1+2+\cdots+n)^2 \text{ by (d)} \\ &= \left(\frac{n(n+1)}{2}\right)^2 \text{ by (b)} \\ &= \frac{n^2(n+1)^2}{4} \text{ by arithmetic } \blacksquare \end{aligned}$$